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AN OPTIMAL OPERATIONAL AVAILABILITY INVENTORY MODEL.(U)

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AN OPTIMAL OPERATIONAL AVAILABILITY INVENTORY MODEL

CACI, Inc.-Federal
Systems and Logistics Division
1815 North Fort Myer Drive
Arlington, Virginia 22209

April 1978

Prepared for:

Ship Support Improvement Project
Naval Sea Systems Command (PMS-306)
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SUMMARY

A mathematical description is given of a computer model, called the Optimal A_o Model, designed to maximize equipment operational availability subject to a budget constraint for spares procurement. Levels are calculated for all items in the equipment parts breakdown and all activities in a multi-echelon supply/repair system. A solution procedure is given based upon the Lagrange multiplier approach with an embedded dynamic programming technique. A description is also given of the Material Flow model designed to calculate parameters of the Optimal A_o Model.

This effort was sponsored by the Ship Support Improvement Project, Naval Sea Systems Command (PMS-306) under contracts N00173-77-C-0184 and N00024-78-C-7020.

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I. INTRODUCTION

In this report, a mathematical description is given of a model which determines inventory levels that maximize equipment availability within budget constraints. The model calculates levels for all items in the top-down breakdown of an equipment and at all levels in a multi-echelon support system. This model represents a continuation of research in a particular area of inventory theory which has become important with respect to operational use in the military services. To provide context, the background of this research and application is summarized in this section.

A. Previous Research

The class of inventory models being considered is characterized as stationary and based upon Markov processes and elements of renewal or queuing theory. For such models, a fixed ordering policy of a simple form (usually an (s,S)-type policy) is chosen for which the inventory level over time becomes a particular stochastic process. The principal problem is to find a stationary distribution of the process which, if it exists, will be a function of the policy used and of the demand distribution, but not of any costs that might be involved. However, an objective function can be imposed upon the process in expressions which may include expected cost per time unit. Values for the parameters that characterize the policy (considered as decision variables) can then be found by standard solution techniques which satisfy the objective function.

Using this approach, Rosenman and Hockstra (10) investigated the inventory control problem for a repairable item in a two-level supply/repair system. In this study, it was assumed that items can be repaired locally (area facilities) or centrally (NICP - National Inventory Control Point) according to given rates, and that losses to the system are negligible. It was further assumed that external demands are Poisson distributed; that lower level facilities use continuous review, one-for-one (S-1,S) ordering policies; and that replenishment times and repair cycles are given constants.

Under these assumptions, a cost-free model was developed with the objective of distributing a given system stock among the various activities in order to minimize total expected customer waiting time.

In the model, stationary distributions for numbers of items in the NICP repair cycle and net (serviceable) stock at the NICP were developed, from which an expression for average delay in satisfying demands from lower activities was derived. Similarly, distributions for repair cycle and stock on-hand at the area facilities were given, from which the average number of back orders were obtained and summed over all areas to find the average customer wait time. Using these results, a marginal value method was used to distribute system stock in order to minimize total customer waiting time.

Sherbrooke (12) also used the stationary process approach to analyze the multi-item problem involving inventories of recoverable (repairable) items in a system of parallel activities (bases) supported by a higher-level activity (depot). The model assumes (S-1,S) continuous review policies at lower activities; since all stock are assumed to be conserved (always repairable), there is no system reordering (procurement) after initial stocks are established. The primary objective of the model is to establish stock levels (values for S) at each activity (including the depot) which minimize the sum of expected back orders on all recoverable items at the lower installations for a given budgetary constraint. Each recoverable item may be repaired at base and/or depot level according to a specified ratio. Economies of scale are not considered, and lateral resupply (redistribution) is assumed not to occur. Repair and shipping times are assumed to be random (implicitly in the assumed form of the demand distribution and utilizing Palm's theorem). Under these assumptions and objective function, a five-step procedure was given for finding optimal solutions.

First, using an expression for expected number of base back orders, the average delay per demand against the depot is found for each item as a function of depot stock. Second, for each level of depot stock and each base, expected base back orders are calculated as a function of the base stock. Third, for each level of depot stock, an allocation to the bases is made which minimizes the total expected back orders. This

is done by a marginal analysis method. Fourth, the minimum expected system back orders are found as a function of total system stock (bases + depot). Finally, the multi-item aspect is considered by the use of a marginal value method to allocate a given investment across items. Each additional increment of investment is assigned to that item for which the largest reduction in expected system back orders will occur.

This model (METRIC - Multi-Echelon Technique for Repairable Item Control) was selected by the Air Force for application in the Advanced Logistics System. A modification of the model (MOD-METRIC) was developed by Muckstadt (6) to consider an equipment (aircraft engine) broken down into a number of subordinate modules. Unlike METRIC, which considered the repairable items to be technically unrelated, the MOD-METRIC model considers relationships between stockages of the equipment as spares and stockages of its subordinate parts. The model is designed to determine levels for both the equipment and modules at bases and depot such that expected base shortages are minimized for a total system investment in spares. A Lagrange multiplier solution technique was used to find the optimal levels. The MOD-METRIC model was used by the Air Force to provision the F-15 weapon system.

The relationships of parts in an equipment was also considered in a model (IOL Optimization Model) developed by General Dynamics (3) and implemented at the Aviation Supply Office for determining shipboard stocks of spares for Navy aircraft. The model considers three indenture levels of the parts breakdown but is confined to one supply echelon (in contrast with the two-echelon representation in METRIC and MOD-METRIC). With assumptions similiar to the other models, the IOL Optimization Model determines stock levels of items at the several indenture levels such that expected back orders are minimized within an overall spares budget constraint. A Lagrange multiplier solution method is used in which stock levels and associated costs are calculated for a grid of multiplier values.

A modification of the METRIC model was developed by the Logistics Management Institute (5) for use in optimally allocating repair dollars and facilities. The model (referred to as METRIC-LMI) contains an insert to METRIC which computes the expected back order reduction for each additional unit of stock for each recoverable

component. Recently the model has been extended to consider hierarchical levels-of-indenture in a manner similiar to that of MOD-METRIC (13). This extension includes a procedure to reduce the computation time required. In the procedure, depot stock at a given indenture is treated as if it were base stock at the next lower indenture, thereby reducing the dimensionality of the computation. Proof of the validity of this approach is given. The procedure is essentially the same as an earlier one developed independently by Fitzgerald (14).

Muckstadt recently has extended the MOD-METRIC model to include more than two supply echelons while retaining the multi-indenturing feature. A three-echelon system is considered in a model for Air Force application (7). In a model developed for application to Navy aircraft engine management (8), four echelons were considered which correspond to organizations with different levels of maintenance capability. Both of these extensions retain the other features of MOD-METRIC and use the same basic solution technique.

The designation of multiple types of repair in a multi-item, multi-location system was considered in a model developed by Porteus and Lansdowne (9). In this model, different repair response times were assumed available with associated costs for support equipment and manning. Like METRIC, the items subject to stockage were assumed independent and not related in terms of a parts breakdown structure. The model minimizes expected shortages within a budget constraint which covers both the procurement of spares and the procurement of equipment and manning levels for the repair facilities. As in previous models, a Lagrangian solution procedure is used.

B. The A_o Inventory Problem

In previous approaches to the inventory problem, analytic models have been formulated which determine stock levels that satisfy some supply-oriented objective such as minimizing expected inventory costs, minimizing expected stockouts, maximizing fill rate, etc. From an operator's point of view, however, the main concern is to keep the equipment in operational use as much of the time as possible. Thus, he is interested in supply policies that minimize the time the equipment is not operational because of lack of spare parts. Referring to the "classical" expression for equipment availability,

$$A_o = MTBF/(MTBF + MTTR + MSRT)$$

where A_o = fraction of time the equipment is operational

MTBF = mean time between failures

MTTR = mean time to repair

MSRT = mean supply response time,

the operator's general objective is to determine stock levels for all repair parts in the equipment such that the mean supply response time is minimized subject to given constraints. It is assumed that the MTBF and MTTR terms are independent of the stockage policy and given as constants.

C. The Optimal A_o Model

In this paper, a model is formulated to satisfy directly the minimum MSRT (maximum A_o) goal. This model is a direct descendant of the models summarized above. In particular, the structure of the model is similar to that of MOD-METRIC in that it considers multi-indentured equipments and a multi-echelon support system. It differs, however, in the form of the objective function and in the solution procedure. In subsequent discussion, this model is referred to as the "Optimal A_o Model."

D. Preliminary Theorems

In the formulation and solution of the Optimal A_o Model, several theorems developed from previous research are used. These theorems are stated (without proofs) for convenient reference.

1. Palm's Theorem (Generalized)

The original form of Palm's theorem, as applied to inventory policies of the (S-1,S) type, states that if demand is Poisson, then the number of units in resupply in the steady state is also Poisson for any distribution of resupply time. The Poisson state probabilities depend on the mean of the resupply distribution, but not on the resupply distribution itself.

This theorem was extended by Feeney and Sherbrooke (2) to include compound Poisson demands. Their theorem is given as follows:

> Let s be the spare stock for an item where demands are compound Poisson with customer arrival rate λ and the resupply time is an arbitrary distribution $\psi(t)$ with mean T. Assume that when a customer is accepted, a resupply time is drawn from $\psi(t)$ that is applicable to all demands placed by that customer. In the backorder case, the steady-state probabilities of x units in resupply are given by the compound Poisson with rate λT; i.e,
>
> $$h(x) \quad = \quad p(x;\ \lambda T) \qquad 0 \leq x < \infty$$
>
> In the lost sales case, under the assumption that a customer is accepted only when the stock on hand, s-x, equals or exceeds the number of demands made by the customer, the steady-state probabilities for the number of units in resupply are
>
> $$h(x) \quad = \quad p(x;\lambda T)\ / \sum_{\omega=0}^{s} p(\omega;\ \lambda T) \qquad 0 \leq x < s$$

2. Convexity Properties of the Backorder Function

The expected number of back orders for a stock level of s and a mean resupply time of T is given by the well-known function,

$$B(s,T) = \sum_{x=s+1}^{\infty} (x-s)\ p\ (x;\ \lambda T)$$

Although convexity properties of this function have been previously given or assumed, a precise statement is given by Porteus and Lansdowne in (9) as follows:

> B(s,T) is (a) for fixed T, strictly decreasing and discretely convex in s, and (b) for fixed s, continuously differentiable, strictly increasing, and strictly convex in T.

3. Generalized Lagrange Multiplier Method

In an important paper by Everett (1), the Lagrange multiplier method is extended to problems of maximizing an arbitrary real valued objective function over any set whatever, subject to bounds of any other finite collection of real valued functions defined on the same set. This result is of particular use in inventory theory where the functions involved are not differentiable because stockages of items are confined to integer values.

In Everett's generalization, the problem is stated as follows:

> Let us suppose that there is a set S (completely arbitrary) that is interpreted as the set of possible strategies or actions. Defined on this strategy set is a real valued function H, called a payoff function. H(x) is interpreted as the payoff (or utility) which accrues from employing the strategy $x \in S$. In addition, there are n real valued functions C^k $(k = 1, \ldots, n)$ defined on S which are called Resource functions. The interpretation of these functions is that employment of the strategy $x \in S$ will require the expenditure of an amount $C^k(x)$ of the kth resource. The problem to be solved is the maximization of the payoff subject to given constraints c^k $(k = 1, \ldots, n)$ on each resource; i.e., to find max H(x), $x \in S$, subject to $C^k(x) \leq c^k$ for all k.

The main theorem given by Everett concerning the use of Lagrange multipliers for this problem is as follows:

Let λ^k ($k = 1, \ldots, n$) be nonnegative real numbers.

Assume $x^* \varepsilon$ S maximizes the function $H(x) - \sum_{k=1}^{n} \lambda^k C^k(x)$ over all $x \varepsilon S$.

Then x^* maximizes $H(x)$ over all those $x \varepsilon S$ such that

$$C^k(x) \leq C^k(x^*) \text{ for all } k.$$

This theorem says, for any choice of nonnegative λ^k ($k = 1, \ldots, n$), if an unconstrained maximum of the new (Lagrangian) function,

$$H(x) - \sum_{k=1}^{n} \lambda^k C^k(x)$$

can be found (where x^*, say, is a strategy that produces the maximum), then this solution is a solution to that constrained maximization problem whose constraints are, in fact, the amount of each resource expended in achieving the unconstrained solution. Thus if x^* produces the unconstrained maximum and required resources $C^k(x^*)$, then x^* itself produces the greatest payoff which can be achieved without using more of any resources than x^* does.

An important corollary to the above theorem is given by Everett as follows:

Let $\{\lambda_1^k\}$, $\{\lambda_2^k\}$, ($k = 1, 2, \ldots, n$) be two sets of λ^k's that produce solutions x_1^* and x_2^* respectively. Furthermore, assume that resource expenditures of these two solutions differ in only the jth resource,

$$C^k(x_1^*) = C^k(x_2^*) \quad \text{for } k \neq j,$$

and that $C^j(x_1^*) > C^j(x_2^*)$. Then

$$\lambda_2^j \geq \left\{H(x_1^*) - H(x_2^*)\right\} / \left\{C^j(x_1^*) - C^j(x_2^*)\right\} \geq \lambda_1^j$$

This theorem states that, given two optimum solutions produced by Lagrange multipliers for which only one resource expenditure differs, the ratio of change in optimum payoff to the change in that resource expenditure is bounded between the two multipliers that correspond to the changed resource.

II. MODEL FORMULATION

In this section, the structure of the Optimal A_o Model is formulated and the objective function is stated. First, however, basic assumptions of the model are listed.

A. Assumptions

The following features and limitations are assumed to apply in the structure of the Optimal A_o Model:

1. Included parts are organized in terms of an equipment with a top-down breakdown that can be represented as an arborescent network similar to the example given in Figure 2-1. Any part may be totally consumable, totally repairable, or any mix thereof.

2. Stocking/maintenance facilities are organized in a hierarchical structure according to supply/maintenance flows which can be represented as an arborescent network as illustrated by the example given in Figure 2-2. Each facility has a colocated maintenance and supply capability. The facility at the top of the structure (usually considered as a manufacturer) is assumed to have an infinite supply of all items. Indenture levels in the support hierarchy are referenced as "echelons" according to usual supply terminology.

3. External demands upon supply are stationary and compound Poisson distributed.

4. All stockage locations use a continuous review, (S-1,S) ordering policy.

5. Mean times to repair are defined to include all equipment down times that are not supply related and are given as constants.

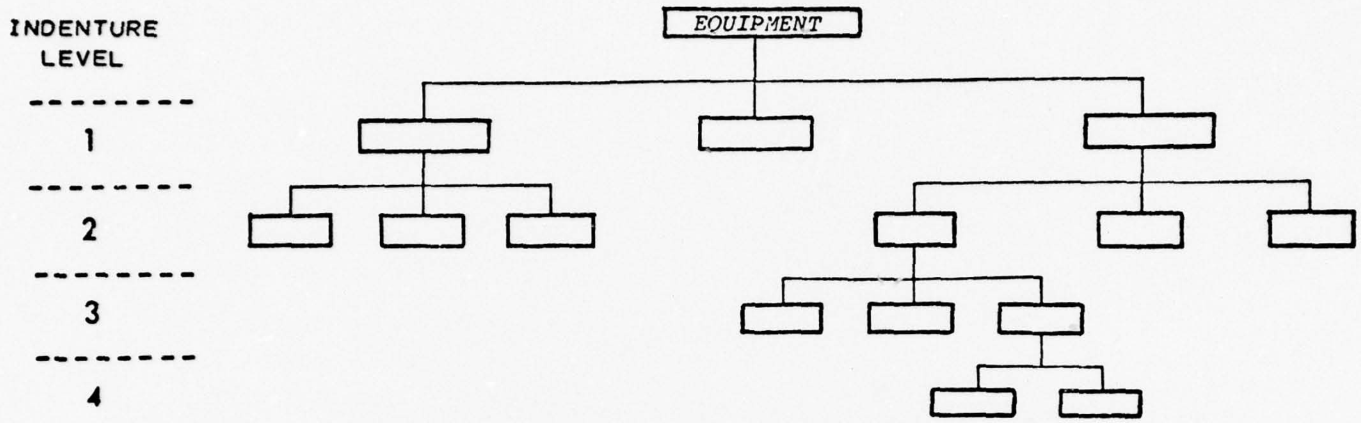

Figure 2-1. Example Parts Hierarchy

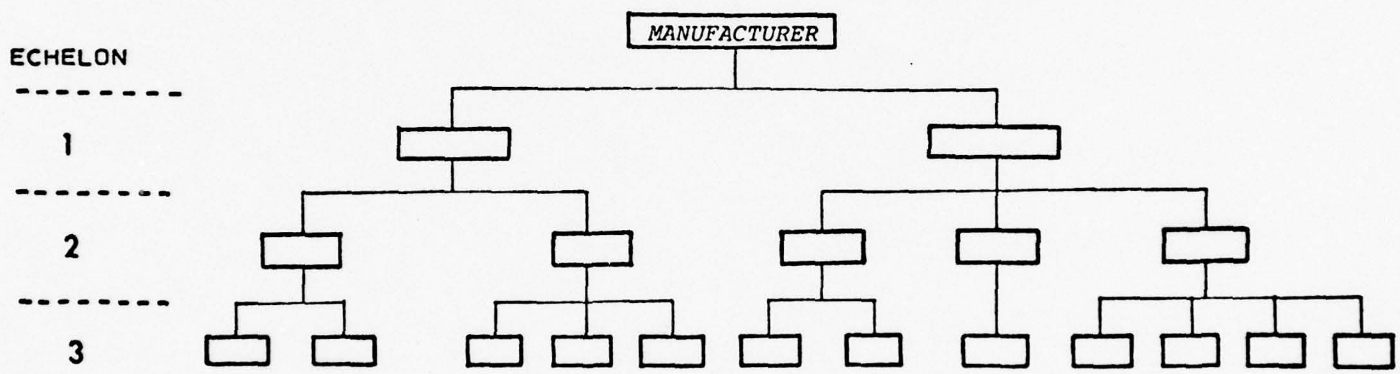

Figure 2-2. Example Support System Hierarchy

The assumed organization of parts in the form of an arborescent network does not preclude the same item appearing in several places within the hierarchy. The model permits a common stock level to be applicable to all appearances and hence to contribute towards reducing mean supply response times in all appearances.

The assumption concerning organization of stocking facilities precludes lateral resupply at a given level of the hierarchy (e.g., ship-to-ship resupply). If, for a given facility in the network, the stocks are physically distributed in several places, it is assumed that the resupply time for direct customers (activities at the next lower indenture level) is independent of such a distribution. Items repaired at any location are assumed (implicity by the arborescence assumption) to be returned to colocated stocks for reissue.

The ordering policy assumption precludes consideration of economies of scale for resupply. In particular, economic order quantities for procurement of consumable items are not allowed. All ordering is on a one-for-one basis (i.e., each time a unit is lost from inventory through discard or being sent to a higher repair facility, a replacement unit is ordered from the next higher supply facility).

In defining the structure of the Optimal A_o Model, a subset of the facilities in the support system is referred to as "user" locations. A user location is a facility that possesses and operates the equipment as well as providing possible supply/maintenance support. Normally (but not necessarily), only activities at the bottom of the support echelon structure are considered as being user locations.

B. Model Structure

The optimal A_o model is defined recursively by considering an arbitrary item in the equipment parts hierarchy and an arbitrary facility in the support system hierarchy. The structure of the model is given by the following definitions and equations:

1. Let i be an arbitrary item in equipment e (which may be e itself).
   Let u = 0 represent an arbitrary facility in the support system and u = 1, 2, ..., U represent facilities at the next lower indenture (i.e., those facilities that submit items for repair directly to or obtain resupply from facility 0).

2. $M_{iu} = D_{iu} + \Upsilon_{iu}$

where M_{iu} = mean time to return a failed unit of item i at location u to a serviceable condition

D_{iu} = expected delay per demand upon inventory for item i at location u

Υ_{iu} = mean time to repair item i at user location u

= 0 if location u does not operate the equipment

3. $D_{iu} = 1/\lambda_{iu} \sum_{x > S_{iu}} (x - S_{iu})\, p\,(x; \lambda_{iu} T_{iu})$ $(u = 0, 1, \ldots , U)$

where S_{iu} = stock level of item i at location u

λ_{iu} = expected number of demands upon inventory for item i at location u

$p(x; \lambda_{iu} T_{iu})$ = probability of x units of stock reduction for item i at location u

T_{iu} = mean resupply time (time to replace an inventory loss) for item i at location u

4. $T_{iu} = \gamma_{iu}(L_{iu} + L'_{iu}) + (1 - \gamma_{iu})(R_{iu} + R'_{iu})$

where γ_{iu} = probability that a demand for item i upon inventory at location u results in a loss (discard or sent elsewhere for repair) which must be replaced through resupply

L_{iu} = average resupply lead time assuming stock availability at the resupply source

L'_{iu} = additional resupply lead time due to expected shortages at the resupply source

R_{iu} = average repair cycle assuming availability of spares for items within i at the next lower indenture level

R'_{iu} = additional repair cycle due to expected shortages of spares for items within i at the next lower indenture level

5. $L'_{iu} = D_{i0}$ $(u = 1, 2, \ldots, U)$

$L'_{i0} = D_{i\nu}$ where ν is the resupply source for location 0

= 0 if location 0 has no resupply (i.e., location 0 is at the top of the support hierarchy)

6. $R'_{iu} = \sum_{j \varepsilon i} \lambda_{ju} M_{ju} / \sum_{j \varepsilon i} \lambda_{ju}$ where j identifies items within i at the next lower indenture level

$= 0$ if i has no subordinate parts

7. $A_{eu} = 1 - \lambda_{eu} M_{eu}$

where A_{eu} = fraction of time equipment e is available for use at location u (defined only for locations u which operate the equipment)

In equation 2, the factor Υ_{iu} represents the marginal mean time to repair item i through replacement from stock or repair of failed subordinate parts at the next lower indenture. Included are all non-supply related functions such as fault isolate, remove and replace, and system checkout. These factors are assumed to be given as constants.

In equation 3, the summation term gives the expected number of back orders for a stock level of S_{iu}. (See Hadley and Whitin (4), section 4-13 for development of this expression.) This can be shown to be equivalent to the expected length of time the stock is in a back order status (Hadley and Whitin, section 1-11). Dividing by the expected number of demands per time unit gives the expected delay in satisfying a demand. The time unit here may be days, months or any other unit that is consistent with units by which delays (resupply time, repair cycle) are measured. Values for λ_{iu} are calculated by the model defined in Section IV.

In equation 4, the factors γ_{iu} are calculated by the model defined in Section IV. The factors L_{iu} and R_{iu} are given constants. The first term (involving resupply lead times) represents losses from stock due to scrap or units sent to higher level repair facilities. The second term represents losses due to amounts cycling through local repair. Although mean values of resupply and repair times are used, these times may in fact be stochastic as long as the conditions of Palm's theorem (generalized) are satisfied. Also, use is made of the fact that the "sum" of compound Poisson distributions is also compound Poisson. (See (11), p. 6-7).

Equation 5 establishes the connection between supply echelons. It states that the additional delay in obtaining resupply is equal to the expected delay per demand upon stocks at the resupply source.

Equation 6 establishes the connection between indenture levels of the parts hierarchy. It states that the additional delay in repairing an assembly is equal to the weighted average of expected delays per demand upon stocks at the next lower indenture level.

Equation 7 gives the operational availability of the equipment in terms of factors defined by previous equations. With proper interpretation of terms, this definition of A_o can easily be translated into that given in Section I.

The above definition of the Optimal A_o Model is recursive on "item" within the parts hierarchy and "location" within the support system hierarchy. If stock levels are given for all items at all locations, a recursive procedure using the equations may be applied to determine corresponding operational availabilities of the equipment at all user locations. The recursion starts with items at the bottom of the parts hierarchy and locations at the top of the support system hierarchy. For such items and locations, additional resupply and repair times (equations 5 and 6) are zero and expected delays can be calculated directly using equations 3 and 4. These delays can then be used in equations 5 and 6 to calculate additional resupply and repair times for the next higher assemblies and next lower locations. Expected delays for these items and locations can then be determined by equations 3 and 4.

This establishes a recursion up the parts hierarchy and down the support system hierarchy until the equipment level is reached at the lowest (user) locations in the support system. Equation 7 is then used to determine the expected equipment availabilities.

C. Objective Function

The model's objective function can be stated as follows:

Find values for $S_{k\nu}$ for all items $k \in e$ and all locations ν in the support system which minimize D_{eu} for all user locations u subject to

$$\sum_{k,\nu} c_k S_{k\nu} = B$$

where

c_k = unit cost of item k

B = given budget for spares procurement

Equations 2 and 7 show that minimizing D_{eu} is equivalent to maximizing A_{eu}, the operational availability of equipment e at user location u.

If the equipment e is not subject to stockage, as is often the case, then S_{eu} in equation 3 and γ_{eu} in equation 4 are set to zero. In this case,

$$D_{eu} = R_{eu} + R'_{eu}$$

The objective function for this case can be rewritten using equation 6 as follows:

Find $S_{k\nu}$ $(k \in e, k \neq e)$ which minimizes

$$\sum_{i \in e} \lambda_{iu} D_{iu} \quad \text{(i at the next lower indenture)}$$

subject to $\sum_{k,\nu} c_k S_{k\nu} = B$

III. SOLUTION PROCEDURE

The optimal solution to the problem defined above is found by a (doubly) recursive procedure based upon equations 2 - 7. First, however, a subproblem is defined and a solution procedure is given for the subproblem.

Substituting equation 4 in 3, the expected delay per demand can be given by

$$D_{i\nu} \quad = \quad D(S_{i\nu}, L'_{i\nu}, R'_{i\nu})$$

where the stock level $S_{i\nu}$, additional resupply time $L_{i\nu}$ and additional repair cycle $R_{i\nu}$ are considered as decision variables for an arbitrary item $i \varepsilon e$ and arbitrary location ν in the support system. Suppose that values for $S_{i\nu}$ are given for all items i and locations ν. The subproblem is to find a particular item and location such that a one unit increase in its stock level will yield the largest decrease in D_{eu} per dollar for some user location u.

The solution of this subproblem is based upon a recognition that the family of functions $D_{i\nu}$ are hierarchically related (by equations 5 and 6), each is a function of three decision variables, and functions at the bottom of the hierarchy depend only upon the stock levels, $S_{i\nu}$. Therefore a dynamic programming solution procedure can be applied as follows:

Define

$$\Delta_S D_{i\nu} \quad = \quad D(S_{i\nu}, L'_{i\nu}, R'_{i\nu}) - D(S_{i\nu}+1, L'_{i\nu}, R'_{i\nu})$$

$$\Delta_L D_{i\nu} \quad = \quad D(S_{i\nu}, L'_{i\nu}, R'_{i\nu}) - D(S_{i\nu}, L^{*'}_{i\nu}, R'_{i\nu})$$

$$\Delta_R D_{i\nu} \quad = \quad D(S_{i\nu}, L'_{i\nu}, R'_{i\nu}) - D(S_{i\nu}, L'_{i\nu}, R^{*'}_{i\nu})$$

where $L^{*'}_{i\nu}$ = least value of $L'_{i\nu}$ obtainable by a unit increase in stock of some part $\omega \varepsilon i$ at the supply source for ν

$R^{*'}_{i\nu}$ = least value of $R'_{i\nu}$ obtainable by a unit increase in stock of some part $r \varepsilon i$ at location ν

Letting ω* represent the part which satisfies $L'^{*}_{i\nu}$ and r* the part which satisfies $R'^{*}_{i\nu}$, find the largest of

$$\text{(a)}\quad \Delta_S D_{i\nu}/c_i \qquad \text{(b)}\quad \Delta_L D_{i\nu}/c_{*} \qquad \text{(c)}\quad \Delta_R D_{i\nu}/c_{r*}$$

and let

$$\begin{aligned} D^*_{i\nu} &= D(S_{i\nu}+1, L'_{i\nu}, R'_{i\nu}) \\ &= D(S_{i\nu}, L'^{*}_{i\nu}, R'_{i\nu}) \\ &= D(S_{i\nu}, L'_{i\nu}, R'^{*}_{i\nu}) \end{aligned}$$

according to which of (a), (b) or (c) is largest, respectively.

With the above definitions and using equations 2, 5 and 6, a recursion across supply echelons and through the parts hierarchy is given by:

$$\begin{aligned} L'^{*}_{i\nu} &= D^*_{i} \quad (\chi = \text{supply source for } \nu) \\ R'^{*}_{i\nu} &= \sum\nolimits_{j \in i-j'} \lambda_{j\nu} M_{j\nu} + \lambda_{j'\nu} M^*_{j'\nu} \Big/ \sum\nolimits_{j \in i} \lambda_{j\nu} \\ M^*_{j\nu} &= D^*_{j\nu} + \Upsilon_{j\nu} \end{aligned}$$

where j identifies parts within i at the next lower indenture, and j' = r* or else j' contains r* as a lower level part.

The recursion is initiated for items i at the bottom of the parts hierarchy and the location ν at the top of the support system hierarchy where $L'_{i\nu}$ and $R'_{i\nu}$ are both zero and hence $D^*_{i\nu} = D(S_{i\nu}+1)$. Justification that this procedure solves the subproblem follows from the convexity properties of the functions $D_{i\nu}$ as established in Section I.

The solution to the orginial problem defined in Section II is given by repeated application of the subproblem. Assume that initial values are given for stock levels for all items and locations. These may all be zero or some minimum value given by policy or current assets. The subproblem and solution procedure is then applied to find the first item and location for which a unit increase in stock will yield the largest

decrease in D_{eu} (or increase in A_{eu}) for some user location u. The stock level of that item and location is then increased by one unit and the subproblem solution procedure again applied. This procedure continues until the budget constraint is first reached

Justification for this solution procedure is based upon Everett's generalization of the Lagrange multiplier method and upon the convexity properties of the functions $D_{i\nu}$ as given in Section I. It has been previously shown that the marginal value approach used does not provide optimal solutions in the general case but approximations are obtained that are sufficiently close to optimality for practical use..

IV. MATERIAL FLOW MODEL

The Material Flow model is designed to calculate values for demand rates, $\lambda_{i\nu}$, and loss probabilities, $\gamma_{i\nu}$, for all items i and locations ν as defined in Section III. The calculations are made based upon failure data, operating factors, location interrelationships, level of repair codings, and several item parameters given as constants.

In the model formulation given below, it is assumed that the support system consists of three echelons (indenture levels in the support system hierarchy) defined as organizational (the lowest echelon), intermediate, and depot (the highest echelon). This formulation can easily be extended to an arbitrary number of echelons.

As a preliminary or setup operation, two factors, E_{eu} and I_{ei}, are defined for use in subsequent equations that determine effective number of failures per month (For convenience, the basic time unit is taken as one month throughout the model formulation.) These factors are defined for user location u, equipment e, and item i as follows:

$$E_{eu} = p_{eu}\sigma_{eu}$$

where E_{eu} = equipment operating unit-months

p_{eu} = number of units of equipment e at location u

σ_{eu} = location operating factor (fraction of calendar time that location u is operational)

$$I_{i_{j+1}ia} = p_{i_j}\sigma_{i_j} I_{i_j ia} \qquad (j = 1, 2, \ldots \text{ until } i_{j+1} = e)$$

$$I_{ei} = \sum_a I_{eia}$$

where I_{ei} = item operating unit-months over all appearances a of item i in equipment e

p_{i_j} = number of units of assembly i_j (containing item i as a subordinate part) in its next higher assembly, i_{j+1}

σ_{i_j} = fraction of time assembly i_j operates when its next higher assembly i_{j+1} operates

The factor I_{ei} for each item in equipment e is calculated recursively by considering successive next higher assemblies that contain item i. The recursion is initiated by setting

$$i_1 = i$$

$$I_{iia} = 1$$

This formulation recognizes the fact that item i may appear in several different places (given by index a) in the equipment and that the item's population and extent of operation over time may differ by appearance.

Next, the expected number of units of item i that generate at location ν for possible repair (at ν or some higher repair location) is given as follows:

$$f_{i\nu} = (730.5 \sum_e E_{e\nu} I_{ei})/B_i \qquad (\nu = u, \text{ user location})$$

$$= 0, \text{ otherwise}$$

$$g_{i\nu} = R_{n\nu} f_{iu}/f_{nu} \, (i \neq e)$$

$$g_{e\nu} = 0 \qquad (\nu \neq u)$$

$$= f_{eu} \qquad (\nu = u)$$

where

f_{iu} = expected number of (real) failures per month of item i at user location u

730.5 = number of hours per month

B_i = mean operating hours between failures (MTBF)

$g_{i\nu}$ = expected number of repairable generations per month of item i at location ν

$R_{n\nu}$ = expected number of repairs per month of assembly n (containing i at the next lower indenture) at location ν (ν = u or contains user location u at some lower indenture of the support hierarchy)

Next, let τ represent the indenture level of a given location ν in the support system, where $\tau = 1$ identifies the organization level, $\tau = 2$ the intermediate level, and $\tau = 3$ the depot level. The following factors are assumed to be given as constants for each item i and level τ :

$k_{i\tau}$ = false removal rate

$d_{i\tau}$ = false removal detection rate

$\alpha_{i\tau}$ = $1 + k_{i\tau}(1 - d_{i\tau})$ = expected number of real failures plus undetected false removals per real failure

ρ_i = fraction of repairs beyond the maintenance capability of the organizational level ($\tau = 1$) which are sent to intermediate level repair ($\tau = 2$) vice depot level repair ($\tau = 3$)

$b_{i\tau}$ = beyond capability of maintenance (BCM) rate ($\tau = 1,2$)

$s_{i\tau}$ = scrap rate

The expected number of units are calculated from the above factors and are given in Table 4-1 according to the indenture level of location ν. They are:

(a) $B^I_{i\nu}$ = units sent by location ν to an intermediate level location for repair

(b) $B^D_{i\nu}$ = units sent by location ν to a depot level location for repair

(c) $S_{i\nu}$ = units scrapped at location ν

(d) $R_{i\nu}$ = units repaired locally at location ν

Finally, values for $\lambda_{i\nu}$ and $\gamma_{i\nu}$ are determined using terms given in Table 4-1

$$\lambda_{i\nu} = B^I_{i\nu} + B^D_{i\nu} + S_{i\nu} + R_{i\nu}$$

$$\gamma_{i\nu} = (B^I_{i\nu} + B^D_{i\nu} + S_{i\nu}) / \lambda_{i\nu}$$

To calculate values for $\lambda_{i\nu}$ and $\gamma_{i\nu}$ according to the above formulation, a recursive procedure is necessary. Since the terms $g_{i\nu}$ are defined in terms of next higher assemblies of items i, the procedure starts with the equipment e at the top of the parts hierarchy where there is no next higher assembly. Also, since terms in Table 4-1 depend upon computations for locations that are lower in the support system hierarchy, the procedure must start at the lowest (organizational) level locations.

Table 4-1. Intermediate Terms of the Material Flow Model

	$B^I_{i\nu}$	$B^D_{i\nu}$	$S_{i\nu}$	$R_{i\nu}$
Organ. ($\tau = 1$)	$b_{i1}\rho_i\alpha_{i1}g_i$	$b_{i1}(1-\rho_i)\alpha_{i1}g_{i\nu}$	$s_{i1}(1-b_{i1})\alpha_{i1}g_{i\nu}$	$(1-s_{i1})(1-b_{i1})\alpha_{i1}g_{i\nu}$
Inter. ($\tau = 2$)		$b_{i2}G^I_{i\nu}$	$s_{i2}(1-b_{i2})G^I_{i\nu}$	$(1-s_{i2})(1-b_{i2})G^I_{i\nu}$
Depot ($\tau = 3$)			$s_{i3}G^D_{i\nu}$	$(1-s_{i3})G^D_{i\nu}$

where $G^I_{i\nu} = \alpha_{i2}g_{i\nu} + b_{i1}\rho_i\alpha_{i1} \sum_{j\in\Pi_{\nu 1}} g_{ij}$

$G^D_{i\nu} = b_{i1}(1-\rho_i)\alpha_{i1} \sum_{j\in\Pi_{\nu 2}} g_{ij} + b_{i2} \sum_{j\in\Pi_{\nu 3}} G^I_{ij}$

$\Pi_{\nu 1}$ = set of all organizational level locations ($\tau = 1$) subordinate (in the support system hierarchy) to (intermediate) location ν

$\Pi_{\nu 2}$ = set of all organizational level locations ($\tau = 1$) subordinate (in the support system hierarchy) to (depot) location ν

$\Pi_{\nu 3}$ = set of all intermediate level locations ($\tau = 2$) subordinate (in the support system heirarchy) to (depot) location ν

APPENDIX A

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